Monte Carlo computation of L_p distance between two densities on $\mathbb{S}^d$

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April 23, 2020

Problem Statement

Let $\mathbb{S}^d = \{\vec{x} \in \mathbb{R}^{d+1} \mid x_1^2 + x_2^2 + \cdots + x_{d+1}^2 = 1\}$ be a unit-norm hypersphere and $\mathcal{P}(\mathbb{S}^d)$ denote a space of densities on $\mathbb{S}^d$. For two densities $f, g \in \mathcal{P}(\mathbb{S}^d)$, we often need the concept of dissimilarity between the two. Unfortunately, even for most well-known distributions on sphere, analytic formula for discrepancy is rarely available, leading to require numerical scheme. We focus on L_p distance between two densities

$$L_p(f, g) = \left(\int_{\mathbb{S}^d} |f(x) - g(x)|^p dx \right)^{1/p} \tag{1}$$

and we combine Monte Carlo integration with importance sampling to approximate (1).

Computation

For notational simplicity, we approximate $L_p(f, g)^p$ and use uniform density $u(x)$ as an importance proposal. Note that $u(x)$ is constant depending on the dimension of a sphere d but we stick to $u(x)$ notation for consistency in the literature of Monte Carlo integration. Then, we have the following,

$$\begin{aligned} L_p(f, g)^p &= \int_{\mathbb{S}^d} |f(x) - g(x)|^p dx \\ &= \int_{\mathbb{S}^d} \frac{|f(x) - g(x)|^p}{u(x)} u(x) dx \\ &= \mathbb{E}_{u(x)} \left[\frac{|f(x) - g(x)|^p}{u(x)} \right] \end{aligned}$$

and as $N \to \infty$, the above can be approximated by

$$\approx \frac{1}{N} \sum_{n=1}^{N} \frac{|f(x_n) - g(x_n)|^p}{u(x_n)} \quad \text{for} \quad x_n \overset{iid}{\sim} u(x).$$