Derivation of Euler-Lagrange Equation for ROF denoising algorithm

Kisung You
kyoustat@gmail.com

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1 Rudin-Osher-Fatemi TV-L2 Denoising Model

Since this article is not meant for comprehensive introduction to the problem, I will not thoroughly review theoretic cues. Depending on which viewpoint one takes upon the problem, a common framework of energy formulation on desirable denoised image u given a noisy picture u_0 is

$$\mathrm{E}[u|u_0] = \lambda \|u - u_0\|_2^2 + \|\nabla u\|_1 \tag{1}$$

where the first term measures closeness of desirable solution to the noisy picture and the second is smoothness penalty - also known as regularization - that penalizes non-smoothness on pursued solution. λ, which originates from Lagrange multiplier, plays a *balancing* role between noise model and image's smoothness requirement. In this formulation, large λ value weighs closeness and small one focuses on smoothing effect. First-order condition for a minimizer u^* is that its *derivative* $\nabla \mathrm{E} = 0$ and this Gâteaux derivative is achieved via Euler-Lagrange equation. A classical solution is to proceed with time-marching method,

$$u_t = -\nabla E[u|u_0] \tag{2}$$

to reach at minimum energy level. For reference, see Rudin, Osher, Fatemi (1992) "*Nonlinear total variation based image removal algorithm*" [1].

2 Basic Calculus

First we need is *divergence theorem.* Suppose Ω is a compact subset of $\mathbb{R}^n$ with a piecewise smooth boundary denoted by $\partial\Omega$. If $\mathbf{F}$ is a continuously differentiable vector field which is defined on Ω, we have

$$\int_\Omega (\nabla \cdot \mathbf{F}) dV = \int_{\partial\Omega} (\mathbf{F} \cdot \mathbf{n}) dS, \tag{3}$$

where left part is volume integral over Ω which expresses, in some sense, change within the entire volume and right side is surface integral over the boundary $\partial\Omega$ of Ω, which somehow measures outwards flux on its surface.

In addition, we need following equality

$$\int_\Omega \nabla \cdot (u \nabla v) = \int_\Omega \nabla u \cdot \nabla v + \int_\Omega u \nabla \cdot (\nabla v) = \int_\Omega \nabla u \cdot \nabla v + \int_\Omega u(\nabla^2 v). \tag{4}$$

3 Derivation of Euler-Lagrange equation

A standard Euler-Lagrange equation pertains to the concept of directional derivative in that we can achieve in via following

$$0 = \nabla \mathrm{E} = \frac{d}{dt}\Big|_{t=0} \mathrm{E}[u+tv] \tag{5}$$

and calculation is as follows.

$$\begin{aligned}
\frac{d}{dt}\mathrm{E}[u+tv] &= \frac{d}{dt}\left(\lambda \int_\Omega (u+tv-u_0)^2 + \int_\Omega |\nabla(u+tv)|\right) \\
&= \lambda \frac{d}{dt}\left(\int_\Omega u^2 + t^2v^2 + u_0^2 + 2tuv - 2tu_0v - 2uu_0\right) \\
&\quad + \frac{d}{dt}\left(\int_\Omega |\nabla(u+tv)|\right) \\
&= \lambda \int_\Omega (2tv^2 + 2uv - 2u_0v) + \frac{d}{dt}\left(\int_\Omega |\nabla(u+tv)|\right).
\end{aligned}$$

For the second term above, using chain rule, we have

$$\frac{d}{dt}\left(\int_\Omega |\nabla(u+tv)|\right) = \int_\Omega \frac{\nabla(u+tv)}{|\nabla(u+tv)|}\cdot \nabla v$$

since

$$\frac{d}{dx}|x| = \frac{x}{|x|} = sgn(x) = \begin{cases} 1 \quad \text{if} & x>0 \\ -1 & x<0 \\ (-1,1) & \text{otherwise.} \end{cases}$$

Combining these two and inserting $t=0$, we have

$$0 = \nabla \mathrm{E} = 2\lambda \int_\Omega ((u-u_0)v) + \int_\Omega \left(\frac{\nabla u}{|\nabla u|}\cdot \nabla v\right) \tag{6}$$

and we need to further apply basic calculus results from the above in that

$$\begin{aligned}
\int_\Omega \left(\frac{\nabla u}{|\nabla u|}\cdot \nabla v\right) &= -\int_\Omega \left(\nabla \cdot \frac{\nabla u}{|\nabla u|}\right) v + \int_\Omega \nabla \cdot v \frac{\nabla u}{|\nabla u|} \\
&= -\int_\Omega \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) v + \int_\Omega \mathrm{div}\left(v \frac{\nabla u}{|\nabla u|}\right) \\
&= -\int_\Omega \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) v + \int_{\partial\Omega} \frac{v}{|\nabla u|}\left(\nabla u \cdot n\right).
\end{aligned}$$

Therefore, if we set $\nabla u \cdot n = 0$, we have

$$0 = \int_\Omega \left((2\lambda(u-u_0)) - \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)\right) v \tag{7}$$

and since it should be satisfied with arbitrary direction v, we finally have

$$\begin{cases} (u-u_0) - \frac{1}{2\lambda}\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$

Then, time-marching methods can be written as following differential equation with Neumann boundary condition,

$$\begin{cases} \frac{\partial u}{\partial t} = (u_0 - u) + \frac{1}{2\lambda}\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

References

[1] Leonid I. Rudin, Stanley Osher, and Emad Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D: Nonlinear Phenomena*, 60(1-4):259–268, November 1992.